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On Quantum Team Games

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Recently Liu and Simaan (2004) convex static multi-team classical games have been introduced. Here they are generalized to both nonconvex, dynamic and quantum games. Puu's incomplete information dynamical systems are modified and applied to Cournot team game. The replicator dynamics of the quantum prisoner's dilemma game is also studied.

KEY WORDS: nonconvex team game; team Cournot; Puu's incomplete information dynamical system; quantum team game; the replicator dynamics of the quantum prisoner's dilemma game.

1. QUANTUM GAMES

Quantum (Orlin Grabbe, 2005) is a generalization of classical games where quantum mechanics techniques are used. It has two main advantages to classical game: The first is the superposition between possible strategies. Thus in the prisoner's dilemma PD case where the classical strategies are cooperate (C) or defect (D) the quantum PD (QPD) admits a strategy of the form aC + bD where a, b are complex constants. The second advantage of quantum games over classical ones is entanglement which can be understood as a kind of communications between the players thus changing the game from noncooperative into cooperative. Mathematically this means that the initial state of the classical PD game can be {CC, CD, DC, DD} i.e., the both players cooperate, or the first cooperate while the second defect or the first defect while the second cooperate or both players defect. In the quantum game the possible states are any linear combination between the above 4-states. Of particular importance are the entangled states which by definition cannot be written as tensor product of the corresponding Hilbert-space of strategies e.g., the state $a|CC\rangle + b|CD\rangle$ can be factored in the form $C \otimes (a|C\rangle + b|D\rangle)$ thus it is not entangled while the state $a|CC\rangle + b|DD\rangle$ is entangled. These entangled states allow more degrees of freedom and allows one to solve some of the dilemmas faced in classical games (Orlin Grabbe, 2005). Since

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we are interested in both evolutionary stable games (Hofbauer and Sigmund, 1998) and Cournot game (Gibbons, 1992) we will review, their quantum formulations. The study of quantum evolutionary game using replicator dynamics (Iqbal and Noor, 2004) for a 2-strategy game begins with the payoff matrix, $\Pi = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]$. Assume the initial state is in the form

$$|\psi_{\text{initial}}\rangle = \sum_{i,j=1}^{2} c_{ij}|ij\rangle, \ \sum_{ij=1}^{2} |c_{ij}|^2 = 1$$

where c_{ij}, i, $j = 1, 2$ are complex constants. The replicator equation thus takes the form

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t = x(1-x)\{&[(a_{11}-a_{21})(|c_{12}|^2 - |c_{22}|^2) + (a_{22}-a_{12})(|c_{21}|^2 - |c_{11}|^2)] \\ &+ x[(a_{21}+a_{12}-a_{11}-a_{22})(|c_{21}|^2 + |c_{12}|^2 - |c_{22}|^2 - |c_{11}|^2)]\} \end{aligned} \quad (1)$$

where x is the fraction of adopters of the strategy 1. This equation is a generalization of the one in (Iqbal and Noor, 2004) for the general payoff matrix Π.

Applying the above formalism to the prisoner's dilemma game PD one gets $\Pi = \left[\begin{smallmatrix} 3 & 0 \\ 5 & 1 \end{smallmatrix}\right]$ and assume the initial state to be the entangled state a|CC⟩ + b|DD⟩ where $|a|^2 + |b|^2 = 1$, one finally gets

$$\mathrm{d}x/\mathrm{d}t = x(1-x)[-x + (2 - 3\,|a|^2)] \quad (2)$$

Now the superiority of quantum games over classical one appears: The equilibrium solutions of (2) are $x = 0, 1, 2/3|a|^2$ (if $1/3 < |a|^2 < 2/3$). The first solution $x = 0$ corresponds to the classical game solution where all players defect. It is asymptotically stable if $1 \geq |a|^2\rangle 2/3$. The second solution $x = 1$ corresponds to the solution of the dilemma where all players cooperate and it is asymptotically stable if $|a|^2 < 1/3$. The mixed solution where some players defect and others cooperate exist and is asymptotically stable if $1/3 < |a|^2 < 2/3$.

It is important to note that the concept of asymptotic stability is necessary for evolutionary stability. Moreover in the case of asymmetric games, it is well defined while evolutionary stability is not (Hofbauer and Sigmund, 1998).

Now Cournot game (Gibbons, 1992) is studied. In this game, the market is controlled by a few number of competing firms producing similar goods. Assume that production decisions are taken at discrete time steps $t = 0, 1, 2, \ldots$ to determine the quantities q_i^t produced by the ith firm at time t. Let c_i be the cost of a unit produced by the ith firm. Assume that the system under consideration consists of two teams each consisting of two firms. The profit functions of the firms are given by

$$\Pi_i^t = q_i^t[a - c_i - bQ^t], \ Q^t = \sum_{i=1}^{2} q_i^t$$

The quantum static $t = 0$ Cournot game has been studied (Li *et al.*, 2002) and the payoff functions are

$$\begin{aligned}\Pi_1^t &= (q_1^t \cosh\gamma + q_2^t \sinh\gamma)(a - c_1 - b\exp(\gamma)Q^t)\\ \Pi_2^t &= (q_2^t \cosh\gamma + q_1^t \sinh\gamma)(a - c_2 - b\exp(\gamma)Q^t)\end{aligned} \tag{3}$$

where γ is the entanglement parameter. In the next section classical games will be generalized to dynamic team games using bounded rationality approach. In Section 3 Puu's incomplete information approach to dynamical games will be introduced, modified and applied to the quantum team Cournot game.

2. CLASSICAL TEAM GAME

Recently Liu and Simaan (2004) convex static multi-team games have been introduced. In these games there are several teams, each team consists of some players.

The important concept is noninferior Nash strategy (NNS) which is Pareto optimal if the players belong to the same team and Nash optimal if they belong to different teams. For the case of games with convex payoffs this is done as follows: Let Π_i^X be the convex payoff of the player i in the team X. Then the team's payoff is given by

$$\Pi^X = \sum_i \omega_i^X \Pi_i^X, \quad 1 \geq \omega_i^X \geq 0, \quad \sum_{\mathrm{i}} \omega_i^X = 1 \tag{4}$$

For the case of continuous games NNS is obtained by

$$\partial\Pi^X / \partial u_i^X = 0$$

where u_i^X is the control parameter of player i in the team X.

This is an important work which is relevant to many realistic systems. However many real systems have nonconvex payoffs e.g., Cournot game (Gibbons, 1992). Consider two teams each consisting of two firms. As an example consider two branches of Kentucky fast food resturants competing against two branches of MacDonalds. The profit functions of the firms are given by

$$\begin{aligned}\Pi_1^{X=1} &= q_1[a - b_1 Q] - c_1 q_1, \quad Q = \sum_{i=1}^{4} q_i\\ \Pi_2^{X=1} &= q_2[a - b_2 Q] - c_2 q_2\\ \Pi_3^{X=2} &= q_3[a - b_3 Q] - c_3 q_3\\ \Pi_4^{X=2} &= q_4[a - b_4 Q] - c_4 q_4\end{aligned} \tag{5}$$

It is clear that the generic form of the profit function $\Pi = q(a - bq)$ is nonconvex hence the previous construction needs to be modified to include static Cournot game.

We propose that players in the same team share some of their payoffs with their teammates. Hence the final payoff of the first player ($i = 1$) of the first team ($X = 1$) is

$$\Pi_1^{\text{Team1}} = \left(1 - \varepsilon_{12}^{X=1} - \varepsilon_{13}^{X=1} - \cdots - \varepsilon_{1,n1}^{X=1}\right)\Pi_1^{X=1} + \varepsilon_{21}^{X=1}\Pi_2^{X=1} + \cdots + \varepsilon_{n1,1}^{X=1}\Pi_{n1}^{X=1}$$

where $1 > \varepsilon_{ij}^X > 0$, $\left(\sum_{\mathrm{i}} \varepsilon_{ij}^X\right) < 1$, $\left(\sum_{\mathrm{j}} \varepsilon_{ij}^X\right) < 1$.

In general the final payoff of the ith player in the team X is

$$\Pi_i^{\text{Team}X} = \left(1 - \sum_{j \neq i} \varepsilon_{ij}^X\right)\Pi_i^X + \sum_{j \neq i} \varepsilon_{ji}^X \Pi_j^X \tag{6}$$

Now to find NNS for nonconvex game one solves the system

$$\partial \Pi_i^{\text{Team}X} / \partial u_i^X = 0.$$

This formulation can be generalized to the dynamical case using Puu's approach (Puu, 1991) with bounded rationality. The corresponding dynamical system for Cournot game becomes

$$\begin{aligned} q_i^{t+1} &= q_i^t + \alpha_i(q_i^t)\partial\Pi_i^{\text{Team1}}/\partial q_i^t, \quad i = 1, 2 \\ q_i^{t+1} &= q_i^t + \alpha_i(q_i^t)\partial\Pi_i^{\text{Team2}}/\partial q_i^t, \quad i = 3, 4 \end{aligned} \tag{7}$$

The factor $\alpha_i(q_i^t)$ in the equations indicates that in the case $\partial\Pi_i/\partial q_i^t > 0$ bigger firms has a greater capacity to increase production (Bischi and Naimzada, 1999).

For simplicity, we begin with one team and take $\alpha_i = \alpha = 1$, $b_i = b$, $i = 1, 2$ hence we get the following system

$$\begin{aligned} q_1^{t+1} &= q_1^t + (1 - \varepsilon_1)\left[a - b.Q - c_1 - bq_i^t\right] - \varepsilon_2 b q_2^t \\ q_2^{t+1} &= q_2^t + (1 - \varepsilon_2)\left[a - b.Q - c_2 - bq_2^t\right] - \varepsilon_1 b q_1^t \end{aligned}$$

The equilibrium solution for the above system is

$$\begin{aligned} q_1 &= [(1 - \varepsilon_2)/b(3 - \varepsilon_1 - \varepsilon_2)]\{a - c_2 + 2(c_2 - c_1)(1 - \varepsilon_1)/(1 - \varepsilon_1 - \varepsilon_2)\} \\ q_2 &= [(1 - \varepsilon_1)/b(3 - \varepsilon_1 - \varepsilon_2)]\{a - c_1 - 2(c_2 - c_1)(1 - \varepsilon_2)/(1 - \varepsilon_1 - \varepsilon_2)\} \end{aligned} \tag{8}$$

which is locally asymptotically stable if both the following conditions are satisfied:

$$2 > b(2 - \varepsilon_1 - \varepsilon_2) > 0, \qquad 4 > b^2(1 - \varepsilon_1 - \varepsilon_2)(3 - \varepsilon_1 - \varepsilon_2). \tag{9}$$

3. PUU'S INCOMPLETE INFORMATION DYNAMICAL SYSTEM

Recently (Puu, 1995) has proposed an alternative to the bounded rationality approach (6) as follows:

$$q_i^{t+1} = q_i^t + \alpha\left(\Pi_i^t - \Pi_i^{t-1}\right)/\left(q_i^t - q_i^{t-1}\right) \tag{10}$$

We propose to call (10) Puu's incomplete information dynamical system. It has a main advantage that it is realistic since a firm does not need to know the form of the profit function to get an estimate of the quantity q_i^t next time step. Instead all what it needs is its profits and the quantities it has produced in the past two time steps. However, it has a serious problem in that the system (10) is numerically unstable as it approach equilibrium. There is no guarantee that the rate of convergence of the profits will be faster than or equal to that of *q(i, t)*. In fact in the case of duopoly this causes serious instabilities of the system (10). Therefore, we propose the following modified system:

$$X_i^t = \left(\Pi_i^t - \Pi_i^{t-1}\right)/\left(q_i^t - q_i^{t-1}\right)$$

$$\text{if } X_i^t > .1q_i^t \text{ then set } X_i^t = .1q_i^t$$

$$\text{if } X_i^t < -.1q_i^t \text{ then set } X_i^t = -.1q_i^t$$

$$q_i^{t+1} = q_i^t + \alpha X_i^t \tag{11}$$

This system allows a change in q_i^t of up to 10% per time step which is both realistic and avoids the singularities of (10).

Now we present Puu's incomplete information dynamical system for Cournot monopoly (i.e., one firm). Assuming $\Pi(t) = q(t)(a - bq(t))$ then (10) becomes

$$q(t+1) = q(t) + \alpha[a - c - b(q(t) + q(t-1))] \tag{12}$$

which is free from singularities. The equilibrium solution is $q = (a - c)/(2b)$ which is locally asymptotically stable if $0 < \alpha b < 3$.

These results shed light on the case of Cournot duopoly (i.e., two firms). In this case there are two possibilities: Either the two firms are different enough such that eventually only one firm persists while the quantity produced by the other tends to zero. In this case, one regains the case of monopoly. The other possibility is that the two firms are close enough that they both persist. In this case the system (11) for duopoly can be approximated by setting $q_1^t \approx q_2^t$ hence we get

$$\begin{aligned} q_1^{t+1} &= q_1^t + \alpha\left[a - c_1 - 2b\left(q_1^t + q_1^{t-1}\right)\right] \\ q_2^{t+1} &= q_2^t + \alpha\left[a - c_2 - 2b\left(q_2^t + q_2^{t-1}\right)\right] \end{aligned} \tag{13}$$

The equilibrium solution of (13) is $((a - c_1)/4b, (a - c_2)/4b)$ and it is locally asymptotically stable if $0 < \alpha b < 3/2$. This does not imply that it is an equilibrium point for (11) but it will be close to its attracting set. Moreover, since a change of up to 10% is allowed in (11) we have:

Proposition 1. *If the system (11) for Cournot duopoly admits an internal solution and if* $0 < \alpha b < 3/2$ *then it has an attracting set which contains the set*

$$\{(a - c_1)(1 \pm .1)/4b, (a - c_2)(1 \pm .1)/4b\} \tag{14}$$

Numerical simulations have agreed with this result.

Now we formulate the dynamic quantum team Cournot game using Puu's incomplete information approach. Consider a team of two players. The team payoff functions are

$$\Pi_1^{\text{Team}}(t) = (1 - \varepsilon_1)\Pi_1^t + \varepsilon_2\Pi_2^t,\ \Pi_2^{\text{Team}}(t) = (1 - \varepsilon_2)\Pi_2^t + \varepsilon_1\Pi_1^t \tag{15}$$

where $\Pi_1^t,\ \Pi_2^t$ are given by (3). The dynamical equations are given by (11). There are two possibilities: Either the two firms are different enough such that eventually only one firm persists while the quantity produced by the other tends to zero. In this case one regains the case of monopoly and there is no more teams. The other possibility is that the two firms are close enough that they both persist. In this case the system (11) for duopoly can be approximated by setting $q_1^t \approx q_2^t$ hence we get the system

$$\begin{aligned} q_1^{t+1} &= q_1^t + \alpha\big\{\exp(\gamma)[(1 - \varepsilon_1)(a - c_1) + \varepsilon_2(a - c_2)] \\ &\quad - 2\exp(2\gamma)\big(q_1^t + q_1^{t-1}\big)(1 - \varepsilon_1 + \varepsilon_2)\big\} \\ q_2^{t+1} &= q_2^t + \alpha\big\{\exp(\gamma)[(1 - \varepsilon_2)(a - c_2) + \varepsilon_1(a - c_1)] \\ &\quad - 2\exp(2\gamma)\big(q_{21}^t + q_2^{t-1}\big)(1 - \varepsilon_2 + \varepsilon_1)\big\} \end{aligned}$$

The equilibrium solution of the above system is $q_1^* = \exp(-g)[(1 - \exp1)(a - c1) + \exp2(a - c2)]/[2(1 - \exp1 + \exp2)]$,

$$\begin{aligned} q_1^* &= \exp(-\gamma)[(1 - \varepsilon_1)(a - c_1) + \varepsilon_2(a - c_2)]/[2(1 - \varepsilon_1 + \varepsilon_2)] \\ q_2^* &= \exp(-\gamma)[(1 - \varepsilon_2)(a - c_2) + \varepsilon_1(a - c_1)]/[2(1 - \varepsilon_2 + \varepsilon_1)] \end{aligned} \tag{16}$$

This solution is asymtotically stable if

$$1 > 2\alpha\exp(\gamma)(1 - \varepsilon_1 + \varepsilon_2) > 0, \quad 1 > 2\alpha\exp(\gamma)(1 - \varepsilon_2 + \varepsilon_1) > 0 \tag{17}$$

Thus we have

Proposition 2. *If the system (11) for the quantum team Cournot game admits an internal solution (16) and if the conditions (17) are satisfied then it has an attracting set which contains the set* $q_1^*(1 \pm .1),\ q_2^*(1 \pm .1)$.

Summarizing, quantum PD game using replicator dynamics is studied. Liu and Simaan (2004) static team game is generalized to quantum dynamic Cournot (nonconvex) game.

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